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EMBEDDING OF A PUNCH IN THE FORM OF AN ELLIPTIC PARABOLOID INTO AN ELASTIC SPATIAL WEDGE†

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Fredholm integral equations of the second kind are obtained and investigated for various boundary conditions on one edge of a wedge which enables one to represent the corresponding Green's functions by integrals of Neumann series in powers of $(1-2\nu)$. Contact problems of the action of a punch in the form of an elliptic paraboloid on an elastic spatial wedge are studied. The asymptotic method of "large λ" [1] is used to solve the integral equations of these problems. The results of the numerical analysis are compared with a well-known contact problem for a half-space [2, 3].

AN EXACT solution of the first fundamental boundary-value problem for an elastic incompressible spatial wedge has previously been obtained [4] using a Kontorovich–Lebedev integral transformation on the real axis and the hypothesis was put forward that, in the case of a Poisson's ratio $\nu \neq 1/2$, the solution of this problem must be obtained in the form of an expansion in powers $(1-2\nu)$. A Kontorovich–Lebedev integral transformation in the complex plane was subsequently used [5, 6] to construct the solution of the first fundamental boundary-value problem in the theory of elasticity for a spatial wedge. In the symmetrical case (sliding support of one face of the wedge), this problem reduces [6] to solving a Fredholm integral equation of the second kind.

1. Using cylindrical coordinates r, φ, z, where the z axis is directed along an edge of the wedge, let us consider a concentrated unit force which acts on the $\varphi=\alpha$ plane of a spatial elastic wedge with an aperture angle α. The other plane of the wedge is assumed either to be free of stresses (problem a) or to be lying without friction on an undeformed foundation (problem b) or to be rigidly fixed (problem c). Here and henceforth, we shall assume for simplicity that the problem is symmetrical with respect to z. The boundary conditions under the assumptions which have been made have the form

$$\varphi=\alpha:\ \tau_{r\varphi}=\tau_{\varphi z}=0;\quad \sigma_\varphi=\delta(r-x)\,\delta(|z|-y) \tag{1.1}$$

$$\begin{aligned}\varphi=0:\ &(\mathrm{a})\ \sigma_\varphi=\tau_{r\varphi}=\tau_{\varphi z}=0\\ &(\mathrm{b})\ v=\tau_{r\varphi}=\tau_{\varphi z}=0\\ &(\mathrm{c})\ u=v=w=0\end{aligned} \tag{1.2}$$

We will express the stresses and displacements in terms of three harmonic functions using well-known formulas [5] and the harmonic functions themselves will be represented by Fourier–Kontorovich–Lebedev integrals in the complex plane. On introducing these representations into (1.1) and (1.2) and applying a number of results of the theory of functions of a complex variable [6], we finally arrive at Fredholm equations of the second kind in the functions $\Phi_m(u)$ $(m=1,2,3,4)$ in terms of which the required displacements $v(r,\alpha,z)$ (the parameter $\beta x>0$) are expressed:

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$$\Phi_m(u) - \theta_m \int_0^\infty L_m(u, y)\Phi_m(y)\,dy = \operatorname{ch}\frac{\pi u}{2} K_{iu}(\beta x), \quad 0 < u < \infty \tag{1.3}$$

$$L_m(u, y) = 2\operatorname{ch}\frac{\pi u}{2}\operatorname{sh}\frac{\pi y}{2} W_m(y) \int_0^\infty \frac{\operatorname{sh}\pi t g_m(t)\,dt}{(\operatorname{ch}\pi t + \operatorname{ch}\pi u)(\operatorname{ch}\pi t + \operatorname{ch}\pi y)}$$

$$\text{(a)}\quad W_1(u) = \frac{\operatorname{ch}\alpha u - \cos\alpha}{\operatorname{sh}\alpha u + u\sin\alpha}, \quad W_2(u) = -\frac{\operatorname{ch}\alpha u + \cos\alpha}{\operatorname{sh}\alpha u - u\sin\alpha}$$

$$g_1(t) = \frac{\operatorname{cth}\alpha t/2}{\operatorname{ch}\alpha t - \cos 2\alpha}, \quad g_2(t) = \frac{\operatorname{th}\alpha t/2}{\operatorname{ch}\alpha t + \cos 2\alpha} \tag{1.4}$$

$$\theta_1 = \theta_2 = (1 - 2\nu)\sin^2\nu$$

$$\text{(b)}\quad W_3(u) = \frac{\operatorname{ch}2\alpha u - \cos 2\alpha}{\operatorname{sh}2\alpha u + u\sin 2\alpha}, \quad g_3(t) = \frac{\operatorname{cth}\alpha t}{\operatorname{ch}2\alpha t - \cos 4\alpha}$$

$$\theta_3 = (1 - 2\nu)\sin^2 2\alpha$$

$$\text{(c)}\quad W_4(u) = \frac{2\varkappa\operatorname{sh}2\alpha u - 2u\sin 2\alpha}{2\varkappa\operatorname{ch}2\alpha u + 2u^2 - 2u^2\cos 2\alpha + \varkappa^2 + 1}, \quad \varkappa = 3 - 4\nu$$

$$g_4(t) = \frac{\operatorname{th}\alpha t\sin^2 2\alpha}{\operatorname{ch}2\alpha t + \cos 4\alpha} - \frac{\sin^2\alpha}{\operatorname{ch}\alpha t}\{g_5(t)[2g_6(t) - tg_7(t)] + \tag{1.5}$$

$$+ g_8(t)[2g_7(t) - tg_6(t)]\}/g_9(t) + 2(1-\nu)\sin\alpha\{g_5(t)\times$$

$$\times(\sin 3\alpha - \sin 2\alpha\operatorname{ch}2\alpha t) - g_6(t)\cos 2\alpha\operatorname{sh}2\alpha t\}/g_9(t)$$

$$g_5(t) = \varkappa\operatorname{sh}2\alpha t\cos 2\alpha - t\sin 2\alpha$$

$$g_6(t) = \cos 2\alpha\operatorname{ch}2\alpha t - \operatorname{ch}3\alpha t - \operatorname{ch}\alpha t\cos 4\alpha \tag{1.6}$$

$$g_7(t) = \sin 2\alpha\operatorname{sh}2\alpha t + \operatorname{sh}\alpha t\sin 4\alpha$$

$$g_8(t) = \sin 2\alpha(\varkappa\operatorname{ch}2\alpha t - 1)$$

$$g_9(t) = [g_5^2(t) + g_8^2(t)](\operatorname{sh}^2 2\alpha t + \cos^2 2\alpha)$$

$$\theta_4 = -(1 - 2\nu)$$

The two Fredholm integral equations (1.3), when $m = 1, 2$, correspond to problem a, while a single integral equation (1.3) with $m = 3$ and $m = 4$ corresponds to problems b and c, respectively. For a fixed $\beta x > 0$, the right-hand side of the integral equation (1.3) does not lie in the space $L_2(0, \infty)$ in which an integral equation similar to (1.3), (1.5) was treated [6], but belongs to a space of functions which are continuously bounded on the semi-axis $C_M(0, \infty)$. We note that the functions $L_m(u, y)$ are of constant sign when $m = 1, 2, 3$, $0 < u$, $y < \infty$, $0 < \alpha < 2\pi$, which facilitates the calculation of the norms in $C_M(0, \infty)$ of the corresponding integral operators in (1.3). The values of these norms, which are equal to $(1 - 2\nu)q_m$, $m = 1, 2$, were found by numerical integration with an accuracy to within 1% for different $\alpha = \pi n/4$:

n	1	2	3	4	5	6
q_1	0.4944	0.4408	0.2936	0.1323	0.1010	0.05234
q_2	1.148	1.301	0.04231	0.02134	0.08913	0.01524

It follows from this in the case of problem a that, when $\alpha = 2/\pi$, for example, the solution of Eq. (1.3) when $m = 2$ can be represented by a Neumann series when the condition $(1 - 2\nu)\ 1.301 < 1$ or $\nu > 0.116$ is satisfied and for any $\nu \in [0, ½]$ when $\alpha = 3\pi/4$. By comparing expressions (1.5) and (1.4) and using the values of q_1 which have been presented above, we arrive at the conclusion that, in the case of problem b for $\alpha = \pi n/8$, such a representation of the solution is also possible for every ν. Estimates and calculations carried out for case c show that, when $\alpha = \pi n/4$ ($n = 1, 2, 3, 5, 6$), $\nu = 0.25$, $\nu = 0.30$ and $\nu = 0.35$, a solution of the integral equation (1.3) can also be constructed by the method of successive approximations. After solving the integral equations (1.3), the displacement $v(r, \alpha, z)$ in the case of problem a, for example, is found using the formula

$$v(r,\alpha,z)=\frac{4}{\pi^3}\frac{1-\nu}{G}\int_0^\infty\int_0^\infty \operatorname{sh}\frac{\pi u}{2}\{W_2(u)\Phi_2(u)-W_1(u)\Phi_1(u)\}\times$$
$$\times K_{iu}(\beta r)\cos\beta z\cos\beta y\,d\beta\,du \tag{1.7}$$

We now turn our attention to the fact that, as follows from (1.3), the functions $\Phi_m(u)$, $m=1,2$ in formula (1.7) are also dependent on βx.

2. Now let a rigid punch which is elliptic in a plan view and with a base which is described by the function $f(r,z)$, which is even with respect to z, be pressed into the plane $\varphi=\alpha$ of the wedge with a force P which is applied on the $z=0$ axis at a distance H from the edge of the wedge. Without any loss of generality, we shall treat the case which is of the greatest interest in applications when the surface of the punch is an elliptic paraboloid, that is,

$$f(r,z)=(r-a)^2/2R_1+z^2/2R_2,\quad R_1<R_2$$

Let us assume that the unknown contact zone is an ellipse Ω: $(r-a)^2/c^2+z^2/b^2=1$, $a>c$. Under the action of the force P, the punch settles by an amount δ and rotates by an angle γ about the line $r=a$. The plane $\varphi=\alpha$ is not subject to loads outside the contact zone. We will neglect the frictional forces between the wedge and the punch. One of the conditions (1.2) is satisfied in the $\varphi=0$ plane. It is required that the distribution of the normal contact stresses under the punch $\sigma_\varphi(r,\alpha,z)=-q(r,z)$ $[(r,z)\in\Omega]$ be found and the quantities a, b, c, δ, γ and H be determined.

The dimensionless parameter λ, which has been introduced here, characterizes the relative remoteness of the contact region from the edge of the wedge. We note that the formulation of the contact problem which has been presented above is not unique and can be modified.

With a knowledge of the displacements of the form of (1.7), the integral equation of this problem can be written as follows:

$$\frac{1-\nu}{G}\frac{2}{\pi^2}\iint_\Omega q(x,y)\,d\Omega\int_0^\infty\int_0^\infty \operatorname{sh}\pi u W_k(u,\beta x)K_{iu}(\beta r)\cos\beta(z-y)\,d\beta\,du=$$
$$=\delta+\gamma(r-a)-\frac{(r-a)^2}{2R_1}-\frac{z^2}{2R_2},\quad (r,z)\in\Omega \tag{2.1}$$
$$W_1(u,\beta x)=\frac{1}{2\operatorname{ch}\pi u/2}\Big(W_1(u)B_1^u\Big\{\operatorname{ch}\frac{\pi y}{2}K_{iy}(\beta x)\Big\}-$$
$$-W_2(u)B_2^u\Big\{\operatorname{ch}\frac{\pi y}{2}K_{iy}(\beta x)\Big\}$$
$$W_{2,3}(u,\beta x)=\frac{W_{3,4}(u)}{\operatorname{ch}\pi u/2}B_{3,4}^u\Big\{\operatorname{ch}\frac{\pi y}{2}K_{iy}(\beta x)\Big\}$$
$$B_m^u=\sum_{n=0}^\infty(\theta_m A_m^u)^n,\quad A_m^u\{f(y)\}=\int_0^\infty L_m(u,y)f(y)\,dy$$

Here A_m^u ($m=1$, 2, 3, 4) are integral operators and the values of $k=1$, 2, 3 correspond to problems a, b and c, respectively. In order to solve the integral equation (2.1), we shall use the asymptotic method of "large λ" [1] which is efficient when the contact region is sufficiently remote from the edge of the wedge. Using well-known theorems, it is possible to prove the validity of the term-by-term integration with respect to u and β of the functional series in the kernel of Eq. (2.1). By using the value of the integral [7, 8]

$$\frac{4}{\pi^2}\int_0^\infty\int_0^\infty \operatorname{ch}\pi u K_{iu}(\beta x)K_{iu}(\beta r)\cos\beta(z-y)\,d\beta\,du=\frac{1}{R}$$
$$R=\sqrt{(r-x)^2+(z-y)^2}$$

we separate out its singular part from the kernel of the integral equation (2.1). This singular part is identical to the kernel of the well-known contact problem for an elastic half-space [2, 3, 9]

$$\frac{1-\nu}{2G}\iint\limits_{\Omega}\frac{q(r,y)}{R}\,dx\,dy+\frac{1-\nu}{2G}\iint\limits_{\Omega}q(x,y)F(x,y,r,z)\,dx\,dy=$$
$$=\pi\left(\delta+\gamma(r-a)-\frac{(r-a)^2}{2R_1}-\frac{z^2}{2R_2}\right),\quad (r,z)\in\Omega \tag{2.2}$$
$$F(x,y,r,z)=\frac{4}{\pi^2}\int_0^\infty\int_0^\infty \operatorname{sh}\pi u\,(W_k(u,\beta x)-\operatorname{cth}\pi u K_{iu}(\beta x))\times$$
$$\times K_{iu}(\beta r)\cos\beta(z-y)\,d\beta\,du$$

Next, we use the following dimensionless quantities and notation (we shall omit the primes):

$$br'=r-a,\quad bx'=x-a,\quad bz'=z,\quad by'=y,\quad b\delta'=\delta,\quad bc'=c,$$
$$bR'=R$$
$$bH'=H,\quad 2Gb^2P'=(1-\nu)\;P,\quad 2Gq'(x',y')=(1-\nu)\,q\,(x,y) \tag{2.3}$$
$$F'(x',y',r',z')=bF(x,y,r,z)$$
$$A=\frac{b}{2R_1},\quad B=\frac{b}{2R_2},\quad \Omega':\frac{r'^2}{c^2}+z'^2=1$$

Lemma. The function $F(x, y, r, z)$ is continuous as well as all its derivatives when (x,y), $(r,z)\in\Omega$. When $\lambda>1+c(1\leqslant\alpha<2\pi)$, $\lambda>\alpha^{-1}+c(c/2\leqslant\alpha\leqslant 1)$, $\lambda>\sqrt{1+c^2(1+\alpha^2)}\,\alpha^{-1}$ $(0<\alpha\leqslant c/2)$, the function $F(x, y, r, z)$ $[(x,y),(r,z)\in\Omega]$ can be represented by the absolutely convergent series

$$F(x,y,r,z)=\sum_{n=1}^{\infty}\frac{f_n(x,y,r,z)}{\lambda^n} \tag{2.4}$$

where $f_n(x, y, r, z)$ are certain polynomials.

Expansion (2.4) is obtained by term-by-term integration of the functional series in the expression for $F(x, y, r, z)$ using known integrals and representations [7, 8]

$$\frac{4}{\pi^2}\int_0^\infty K_{iu}(\beta x)K_{iu}(\beta r)\cos\beta(z-y)\,d\beta=\frac{1}{\sqrt{xr}\operatorname{ch}\pi u}P_{iu-1/2}\left(1+\frac{R^2}{2rx}\right)$$
$$P_{iu-1/2}(1+\theta^2)=F\left(\frac{1}{2}-iu,\frac{1}{2}+iu,1;-\frac{\theta^2}{2}\right) \tag{2.5}$$
$$x\int_0^\infty K_{ig}(\beta x)K_{iu}(\beta x)\cos\beta(z-y)\,d\beta=\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-y}{x}\right)^{2n}\frac{\pi^2}{2^{2n+1}}\times$$
$$\times\frac{1}{[(2n)!]^2(\operatorname{ch}\pi u+\operatorname{ch}\pi g)}\times$$
$$\times\begin{cases}1, & n=0\\ \prod\limits_{k=0}^{n-1}[(1+2k)^2+(u+g)^2][(1+2k)^2+(u-g)^2], & n=1,2,\ldots\end{cases}\quad\left|\frac{z-y}{x}\right|<2$$

and expressions similar to them. In formulas (2.5), $F(a, b, c; x)$ is a hypergeometric Gaussian function which is expanded in series. Terms in which r or x occur in the denominator are expanded in Taylor series of powers of r/λ or x/λ. Since all the series converge absolutely for the λ indicated in the lemma, they converge for any order of summation and they can be regrouped in the form of (2.4). Here, it is easy to determine the explicit form of the functions $f_n(x, y, r, z)$, $n = 1, 2, \ldots$.

Now, by expanding the solution of integral equation (2.2), (2.4) in the form

$$q(x,y)=\sum_{n=0}^{\infty}\frac{q_n(x,y)}{\lambda^n} \tag{2.6}$$

substituting (2.6) into (2.2) and (2.4) and equating terms of similar powers of λ, we obtain an infinite system of integral equations in $q_n(x, y)$ $(n = 1, 2, \ldots)$.

$$
\begin{aligned}
&1.\quad \iint_\Omega \frac{q_0(x,y)}{R}\,dx\,dy = \pi(\delta + \gamma r - Ar^2 - Bz^2)\\
&2.\quad \iint_\Omega \frac{q_1(x,y)}{R}\,dx\,dy = -(a_0 + \varkappa_1)\iint_\Omega q_0(x,y)\,dx\,dy\\
&3.\quad \iint_\Omega \frac{q_2(x,y)}{R}\,dx\,dy = \iint_\Omega \Big[(x+r)\Big(\frac{a_0}{2} + \varkappa_2^{10}\Big)q_0(x,y) +\\
&\qquad\qquad + (a_0 - \varkappa_1)\,q_1(x,y)\Big]\,dx\,dy\\
&4.\quad \iint_\Omega \frac{q_3(x,y)}{R}\,dx\,dy = -\iint_\Omega \Big[\Big((x^2+r^2)\Big(\frac{3}{8}a_0 + \varkappa_3^{20}\Big) +\\
&\qquad\qquad + rx\Big(\frac{a_0}{4} + 2\varkappa_3^{11}\Big) + \frac{a_1}{2}R^2 - \varkappa_3(z-y)^2\Big)q_0(x,y) -\\
&\qquad\qquad - (x+r)\Big(\frac{a_0}{2} + \varkappa_2^{10}\Big)q_1(x,y) + (a_0 + \varkappa_1)\,q_2(x,y)\Big]\,dx\,dy
\end{aligned}
\tag{2.7}
$$

etc., where

$$
\begin{gathered}
(r, z) \in \Omega, \quad a_0 = \int_0^\infty (\operatorname{th} \pi u W^k(u) - 1)\,du\\
a_1 = -\int_0^\infty (\operatorname{th} \pi u W^k(u) - 1)\Big(\frac{1}{4} + u^2\Big)\frac{du}{2}, \quad k = 1, 2, 3\\
W^1(u) = W_1(u) - W_2(u)\\
W^k(u) = W_k(u), \quad k = 2, 3
\end{gathered}
\tag{2.8}
$$

$$
\varkappa_i^{jl} = \frac{4}{\pi^2}\int_0^\infty \operatorname{sh}\frac{\pi u}{2}\,[W_1(u)\,B_1^u\{J_i^{j}(u, y)\} - W_2(u)\,B_2^u\{J_i^{jl}(u, y)\}]\,du, \quad k = 1
$$

$$
\varkappa_i^{jl} = \frac{4}{\pi^2}\int_0^\infty \operatorname{sh}\frac{\pi u}{2}\,W_k(u)\,B_k^u\{J_i^{jl}(u, y)\}\,du, \quad k = 2, 3, \quad \varkappa_i^{00} = \varkappa_i
$$

$$
J_i^{jl}(u, y) = \int_0^\infty\int_0^\infty \frac{\cos us \cos yt \operatorname{ch}^j s \operatorname{ch}^l t}{(\operatorname{ch} s + \operatorname{ch} t)^i}\,ds\,dt, \quad j + l < i
$$

The values of the constants, calculated using formulas (2.8) when $\alpha = \pi n/4$ for problems a, b and c $(k = 1, 2, 3)$ and $\nu = 0, 3$ are shown in Table 1.

In the successive solution of Eqs (2.7), it can be shown that their right-hand sides are always polynomials in r and z and their solution can therefore be found in closed form using the formulas in Sec. 52 of [9]. Here, each function $q_n(x, y)$ $(n = 0, 1, \ldots)$ has a root singularity on the boundary $\partial\Omega$ of the ellipse of contact. It is necessary to set up the condition $q(x, y) = 0$, $(x, y) \in \Omega$ by virtue of the smoothness of the selected shape of the base of the punch. It follows from the results in [2] that the existence of such a solution which is bounded on $\partial\Omega$ in the given formulation of the problem will depend on the number of terms which are retained in the expansion (2.6) and can be constructed in the case being considered here if, in (2.6), we confine ourselves to an accuracy of up to $O(\lambda^{-4})$. By invoking the integral conditions for the equilibrium of the punch

$$
\iint_\Omega q(x, y)\,dx\,dy = P, \quad \iint_\Omega q(x, y)\,x\,dx\,dy = P(H - \lambda)
\tag{2.9}
$$

and introducing notation by means of the formulas

TABLE 1

n	a_0	a_1	$\varkappa_1$	$\varkappa_2{}^{10}$	$\varkappa_3{}^{20}$	$\varkappa_3{}^{11}$	$\varkappa_3$
				Problem *a*			
1	21.16	−10.26	27.56	14.39	10.40	7.428	7.050
2	1.378	−0.4499	8.224	4.847	4.932	1.976	1.620
3	0.07944	−0.01992	0.4103	0.2532	0.2620	0.09904	0.09262
				Problem *b*			
1	−0.4008	0.8749	1.140	0.7311	0.6741	0.3048	0.3570
3	-7.452×10^{-3}	-6.022×10^{-3}	0.04076	0.02465	0.02176	0.01082	0.0180
5	−0.2011	0.03358	6.429×10^{-3}	3.640×10^{-3}	2.376×10^{-3}	1.880×10^{-3}	2.286×10^{-3}
				Problem *c*			
1	−1.561	2.308	−0.02224	−0.01112	0.02723	−0.01703	−0.03080
2	−0.5222	0.1355	0.01677	8.383×10^{-3}	-6.449×10^{-4}	6.331×10^{-3}	9.500×10^{-3}
5	−0.3010	0.04930	-4.778×10^{-3}	-2.389×10^{-3}	-5.400×10^{-4}	-1.585×10^{-3}	-2.134×10^{-3}

$$S_{00}=K,\quad S_{01}=\frac{E-c^2K}{c^2(1-c^2)},\quad S_{10}=\frac{K-E}{1-c^2}$$

$$S_{11}=\frac{(1+c^2)E-2c^2K}{3c^2(1-c^2)^2},\quad S_{02}=\frac{2(1-2c^2)E+c^2(3c^2-1)K}{3c^4(1-c^2)^2}$$

$$S_{20}=\frac{(3-c^2)K-2(2-c^2)E}{3(1-c^2)^2},\quad f_0=3c^3(S^2{}_{11}-S_{02}S_{20})$$

$$f_1=(2S_{20}-c^2S_{11})/f_0$$

$$f_2=(2S_{11}-c^2S_{02})/f_0$$

$$f_3=(2c^2S_{02}-S_{11})/f_0$$

$$f_4=(2c^2S_{11}-S_{20})/f_0 \tag{2.10}$$

$$f_5=\frac{(a_0+2\varkappa_2{}^{10})((a_0+\varkappa_1)S_0^{-1}\lambda^{-3}-\lambda^{-2})}{1+(a_1-a_0/4-2\varkappa_3{}^{11})/(3\lambda^3S_{01})}$$

$$f_6=\frac{a_0/2+\varkappa_2{}^{10}}{6S_{00}S_{01}}f_5-\frac{a_0{}^2-\varkappa_1{}^2}{S_{00}{}^2}$$

$$f_7=\frac{f_6}{\lambda^2}-\frac{a_0+\varkappa_1}{\lambda S_{00}}\left(1+\frac{f_6}{\lambda^2}\right)-\frac{cS_{10}}{2\lambda^3S_{00}}((a_1-2\varkappa_3)(c^2f_3-f_2)+$$

$$+2\varepsilon_1(f_1-c^2f_4)),\quad \varepsilon_1=-(3a_0/8+a_1/2+\varkappa_3{}^{20})$$

$$f_8=1+f_7+\frac{\varepsilon_1c^2}{3\lambda^3S_{00}}-\frac{a_1}{6\lambda^3S_{00}}$$

$$f_9=c^2\left(\frac{f_7}{5}+\frac{\varepsilon_1c^2-a_1/6}{5\lambda^3S_{00}}\right)$$

$$f_{10}=\frac{f_7}{3}+\frac{\varepsilon_1c^2/3-a_1/2}{5\lambda^3S_{00}}$$

$$f_{11} = \frac{3S_{00}}{\pi f_8}$$
$$f_{12} = (a_1 - 2\varkappa_3) f_3 - 2\varepsilon_1 f_4$$
$$f_{13} = 2\varepsilon_1 f_3 - (a_1 - 2\varkappa_3) f_2$$
$$f_{14} = -\frac{\pi c f_{12}}{3\lambda^3 S_{00}}$$
$$f_{15} = -\frac{2\pi c}{3}\left[-f_4 + \frac{f_{12}c}{\lambda^3}\left(-\frac{f_1 S_{10}}{2S_{00}} + \frac{c^2 f_4 S_{01}}{2S_{00}} + \frac{f_1 - f_4}{3}\right)\right]$$
$$f_{16} = -\frac{2\pi c}{3}\left[f_3 + \frac{f_{12}c}{\lambda^3}\left(\frac{f_2 S_{10}}{2S_{00}} - \frac{c^2 f_3 S_{01}}{2S_{00}} - \frac{f_2 - f_3}{3}\right)\right]$$
$$f_{17} = \frac{2cS_{00}}{f_8}\left[f_4\left(f_{10} - \frac{c^2 S_{01}}{2S_{00}} f_8\right) - \frac{f_1}{c^2}\left(f_9 - \frac{c^2 S_{10}}{2S_{00}} f_8\right)\right]$$
$$f_{18} = \frac{2cS_{00}}{f_8}\left[\frac{f_2}{c^2}\left(f_9 - \frac{c^2 S_{10}}{2S_{00}} f_8\right) - f_3\left(f_{10} - \frac{c^2 S_{01}}{2S_{00}} f_8\right)\right]$$
$$f_{19} = f_1 + f_4 + \frac{c(f_{13} - f_{12})}{\lambda^3}\left[f_1\left(\frac{1}{3} - \frac{S_{10}}{2S_{00}}\right) - f_4\left(\frac{1}{3} - \frac{c^2 S_{01}}{2S_{00}}\right)\right]$$
$$f_{20} = f_2 + f_3 + \frac{c(f_{13} - f_{12})}{\lambda^3}\left[f_2\left(\frac{1}{3} - \frac{S_{10}}{2S_{00}}\right) - f_3\left(\frac{1}{3} - \frac{c^2 S_{01}}{2S_{00}}\right)\right]$$

where $E = E[\sqrt{(1-c^2)}]$, $K = K[\sqrt{(1-c^2)}]$ are complete elliptic integrals, we finally obtain

$$q(x, y) = \frac{3P}{2\pi c}\sqrt{1 - \frac{x^2}{c^2} - y^2}, \quad H = \lambda \tag{2.11}$$

$$\frac{B}{A} = \frac{2S_{00} f_{19}(1 - f_{11} f_{14}) - (f_{12} - f_{13})(f_{11} f_{15} + f_{17})/\lambda^3}{2S_{01} f_{20}(1 - f_{11} f_{14}) - (f_{12} - f_{13})(f_{11} f_{16} + f_{18})/\lambda^3} \tag{2.12}$$

$$\frac{P}{B} = \frac{(f_{11} f_{14} + f_{15}) A/B + f_{11} f_{18} + f_{16}}{1 - f_{11} f_{14}} \tag{2.13}$$

$$\frac{\delta}{B} = \frac{(f_{11} f_{15} + f_1) A/B + f_{11} f_{16} + f_{18}}{1 - f_{11} f_{14}} \tag{2.14}$$

$$\frac{\gamma}{B} = f_5\left[\frac{\delta}{2S_{00}B} + c\left(\frac{A}{B} f_1 - f_2\right)\left(\frac{1}{3} - \frac{S_{10}}{2S_{00}}\right) + c\left(f_3 - \frac{A}{B} f_4\right)\left(\frac{1}{3} - \frac{c^2 S_{01}}{2S_{00}}\right)\right] \tag{2.15}$$

Formulas (2.11)–(2.15) determine the solution of the problem in question with an accuracy up to $O(\lambda^{-4})$. Equation (2.12) served for the determination of c or the eccentricity of the contact ellipse $e = \sqrt{1 - c^2}$. It is then possible to find b from Eq. (2.13) and this means also the values of a and b [dimensional see (2.3)]. The extent of the embedding and the skewness of the punch are determined from (2.14) and (2.15), respectively.

When $\lambda \to \infty$, the solution (2.11)–(2.15) reduces to the well-known solution of Lur'ye [3] of a contact problem on the embedding of an elliptic paraboloid into an elastic half-space.

3. Let us carry out a numerical analysis of the solution (2.11)–(2.15), taking problem a with $\alpha = \pi/2$, $\nu = 0.3$ as an example (it is analogous for other values of α).

A plot of relation (2.12), which relates the ratio of the semi-axes of the contact ellipse c to the ratio of the radii of curvature of the punch $R_1/R_2 = B/A$ when $\lambda = 2$ and $\lambda = \infty$ is shown in Fig. 1. The calculations show that, when $0.1 \leqslant c \leqslant 0.9$, the difference between the corresponding values of R_1/R_2 for $\lambda = 10$ and $\lambda = \infty$ does not exceed 0.1%. The quantity δ/B from (2.14) when $\lambda = \infty$ tends extremely slowly to its limiting value and, for example, when $\lambda = 10$, $c = 0.1$, differs from it by 15%. Hence, the closeness of the contact region to the edge of the wedge has a far greater effect on the extent of embedding of the punch than on the eccentricity of the contact ellipse. The quantity γ/B is negative and of order of magnitude 10^{-2} when $\lambda = 4$, $0.1 \leqslant c \leqslant 0.9$ and $\gamma/B \to 0$ when $\lambda \to \infty$ [3]. If follows from (2.3), (2.13) and (2.15) that the quantity δ is proportional to $P^{2/3}$ (here,

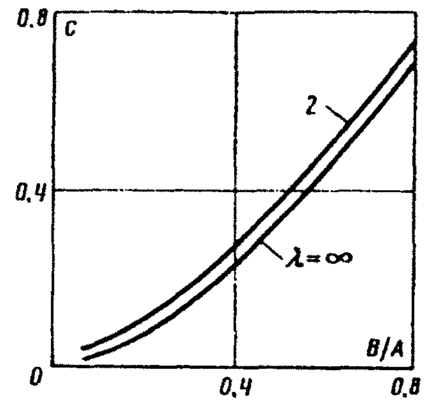

FIG. 1.

δ and P are dimensional) [3]. It is seen from (2.3), (2.13) and (2.15) that the angle γ is proportional to the dimensional magnitude of $P^{1/3}$.

Plots of the dependence of the quantities

$$\alpha_1 = \frac{\delta}{P^{2/3}} \frac{2R_2^{1/3} G^{2/3}}{(1-\nu)^{2/3}}, \quad \alpha_2 = -\frac{\gamma}{P^{1/3}} \frac{2R_2^{2/3} G^{1/3}}{(1-\nu)^{1/3}}$$

(δ and P are dimensional here) on c are shown for different λ in Fig. 2 by the solid and dashed lines, respectively. It is seen that, for certain (not very large) λ, the embedding of the punch can become larger as the shape of the contact region approaches a circular shape.

Formulas (2.11) show that the force is applied at the centre of the contact ellipse and that the distribution of the contact stresses is symmetrical about the axes of symmetry of this ellipse. It turns out that the terms which take account of the asymmetry of the function $q(x, y)$ and of the point of application of the force have an order of magnitude of λ^{-4}. In order to take account of them, that is, to solve integral equation (2.2) with an accuract of $O(\lambda^{-5})$ it is possible to specify only one of the quantities R_1 or R_2 and, here, R_1/R_2 will be determined during the course of the construction of a bounded solution of the type (2.11).

The values of the quantity $\alpha_3 = \frac{2}{3}\pi q(x, 0)/P$, calculated using (2.11) at several points of the contact region for $R_1/R_2 = 0.5$ and for different λ are given below.

x	0	0.2	0.4	0.6
α_3 λ = 2, c = 0.6230	1.605	1.520	1.231	0.4321
α_3 λ = 4, c = 0.6296	1.588	1.506	1.227	0.4813
α_3 λ = ∞, c = 0.6306	1.586	1.504	1.226	0.4880

In conclusion, we note that the method of "large λ" only enables one, within the framework of a reasonable formulation of the problem and subject to the condition $q(x, y) = 0$, $[(x, y) \in \partial\Omega]$, to determine the contact

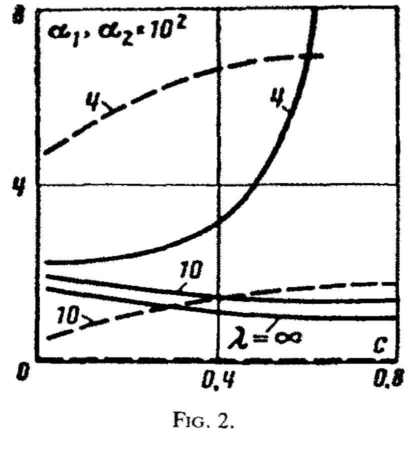

FIG. 2.

ellipse and other quantities to a limited degree of accuracy. In the case of a punch with planar base, when the solution of (2.2) is sought in a class of functions having a root singularity on $\partial\Omega$, all the required quantities can be found to any degree of accuracy (in this case, the contact ellipse is assumed to be known).

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AN EFFECTIVE METHOD OF VERIFYING HADAMARD'S CONDITION FOR A NON-LINEARLY ELASTIC COMPRESSIBLE MEDIUM†

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A new effective criterion is proposed for the validity of Hadamard's condition in a non-linearly elastic compressible body. The verification of Hadamard's condition reduces to analysing a simply structured system of inequalities, so that its validity can be investigated by analytical means, using the same technique for all compressible materials.

INTRODUCTION

It has been shown [1] that for an isotropic incompressible material Hadamard's condition, according to which the velocities of propagation of plane waves of small amplitude in a uniformly stressed elastic medium must be real [2, 3], is equivalent to a system of nine elementary inequalities.

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